Quantum Doubles From A Class Of Noncocommutative Weak Hopf Algebras

Fang Li *
Department of Mathematics, Zhejiang University
Hangzhou, Zhejiang 310028, China

Yao-Zhong Zhang †
Department of Mathematics, University of Queensland
Brisbane, Qld 4072, Australia

February 1, 2008

Abstract

The concept of biperfect (noncocommutative) weak Hopf algebras is introduced and their properties are discussed. A new type of quasi-bicrossed products are constructed by means of weak Hopf skew-pairs of the weak Hopf algebras which are generalizations of the Hopf pairs introduced by Takeuchi. As a special case, the quantum double of a finite dimensional biperfect (noncocommutative) weak Hopf algebra is built. Examples of quantum doubles from a Clifford monoid as well as a noncommutative and noncocommutative weak Hopf algebra are given, generalizing quantum doubles from a group and a noncommutative and noncocommutative Hopf algebra, respectively. Moreover, some characterisations of quantum doubles of finite dimensional biperfect weak Hopf algebras are obtained.

I Introduction

In a recent work [1], quantum doubles of finite dimensional Hopf algebras and finite groups are generalized by one of the authors to those of certain finite dimensional weak Hopf algebras and finite Clifford monoids so as to obtain singular solutions of the quantum Yang-Baxter equation. However, the procedure in [1] is not suitable for noncocommutative weak Hopf algebras. So, it is interesting to construct quantum doubles of noncocommutative weak Hopf algebras. The aim of this paper is to give one class of noncocommutative weak Hopf algebras from which their quantum doubles can be obtained.

As is known [2], bicrossed product is a fundamental tool to construct the quantum double of a Hopf algebra. Quasi-bicrossed product plays a similar role in [1] for the construction of quantum doubles of certain weak Hopf algebras (in particular, finite Clifford monoids). In [3], the concept of weak Hopf pairs was introduced as a generalization of the Hopf pairs of Takeuchi [4]. Using the weak Hopf skew-pairs, one type of quasi-bicrossed products, which lie between general quasi-bicrossed

*fangli@zju.edu.cn
†yzz@maths.uq.edu.au

products and quantum quasi-doubles, were constructed when one of the two weak Hopf algebras in the product is cocommutative. In section III of this paper, we will generalize the results and construct quasi-bicrossed products of the weak Hopf skew-pairs corresponding to the case where both weak Hopf algebras in the product are non-cocommutative (see Theorem III.6 below).

A bialgebra H over a field k is called a *weak Hopf algebra* [1] if there exists $T \in Hom_k(H,H)$ such that $id * T * id = id$ and $T * id * T = T$ where $*$ is the covolution product in $Hom_k(H,H)$; T is called a *weak antipode* of H. Weak Hopf algebras lie between left (resp. right) Hopf algebras and bialgebras. So far, two types of such weak Hopf algebras have been found, which are the monoid algebra kS of a regular monoid S [1] and the almost quantum algebra $wsl_q(2)$ [5] (see also [6] for weak Hopf algebras corresponding to $U_q[sl_n]$).

An application of weak Hopf algebras was found in the construction of non-invertible solutions of the (quantum) Yang-Baxter equation in [1, 5]. It was found that for a finite dimensional cocommutative perfect weak Hopf algebra H with an invertible weak antipode T, the quasi-bicrossed product $H^{op*} \infty H$ (which is called the *quantum double* of H, denoted by $D(H)$), is a quasi-braided almost bialgebra equipped with the quasi-R-matrix $R = \sum_{i=1}^{n}(1 \infty e_i) \otimes (e^i \infty 1) \in D(H) \otimes D(H)$ where $\{e_i\}_{i=1}^n$ is a basis of H as a vector space and $\{e^i\}_{i=1}^n$ is its dual basis in H^{op*}. Then, R is a solution of the quantum Yang-Baxter equation. In [7], it was shown that this solution R is von Neumann regular but not invertible in general. An example of this solution was constructed from the cocommutative perfect weak Hopf algebra $H = kS$ for any finite Clifford monoid S.

Although the quantum double of a finite Clifford monoid is indeed a generalization of the quantum double of a finite group, the quantum doubles in [1] can not usually be regarded as generalizations of quantum doubles of Hopf algebras due to the cocommutativity of the weak Hopf algebras considered in [1]. The goal of this paper is to overcome this so as to construct quantum doubles of noncocommutative weak Hopf algebras. We will give one class of noncocommutative weak Hopf algebras from which their quantum doubles can be obtained. Firstly, we introduce the concept of biperfect weak Hopf algebras and discuss their properties. Then we construct a new type of quasi-bicrossed products by means of the weak Hopf skew-pairs of the weak Hopf algebras which are generalizations of the Hopf pairs introduced by Takeuchi [3]. As a special case, the quantum double of a finite dimensional biperfect (noncocommutative) weak Hopf algebra is built. Examples of quantum doubles from a Clifford monoid and a noncommutative and noncocommutative weak Hopf algebra are given as generalizations of those from a group and a noncommutative and noncocommutative Hopf algebra, respectively. Moreover, we discuss some characterisations of quantum doubles of finite dimensional biperfect weak Hopf algebras.

II Preliminaries

Throughout the paper, k stands for a field. Some notations and definitions unexplained here can be found in [8], [2], [1] and [9]. The word "quantum quasi-double" of a weak Hopf algebra in [1] will always be replaced with "quantum double".

We recall [1] that a linear space H is a k-*almost bialgebra* if (H, μ, η) is a k-algebra and (H, Δ, ε) is a k-coalgebra with $\Delta(xy) = \Delta(x)\Delta(y)$ for $x, y \in H$. If K is a subalgebra and also a sub-coalgebra of H, then K itself is an almost bialgebra, called as an *almost sub-bialgebra* of H.

Combining formally with the definition of the weak Hopf algebras, we say in [7] that an almost bialgebra H is an *almost weak Hopf algebra* if there exists $T \in Hom_k(H,H)$ such that $id * T * id = id$ and $T * id * T = T$, where T is called an *almost weak antipode* of H.

Let H be an almost bialgebra. If there exists an $R \in H \otimes H$ such that for all $x \in H$, $\Delta^{op}(x)R = R\Delta(x)$, then R is called a *universal quasi-R-matrix*; if simultaneously, $(\Delta \otimes id)(R) = R_{13}R_{23}$ and $(id \otimes \Delta)(R) = R_{13}R_{12}$ are satisfied, then we call H a *quasi-braided almost bialgebra* with a *quasi-R-matrix* R (see [1]). Moreover, if H is a bialgebra and R is invertible, then H is called a *braided bialgebra*.

Let H be a bialgebra and C a coalgebra. If C is a left H-module and $\Delta(hc) = \Delta(h)\Delta(c)$ for every $h \in H$ and $c \in C$, then we call the coalgebra C a *left quasi-module-coalgebra* over H. Moreover, if $\varepsilon(hc) = \varepsilon(h)\varepsilon(c)$, then C is called a *left module-coalgebra* over H. Right quasi-module-coalgebra and right module-coalgebra can be defined similarly.

A pair (X, A) of bialgebras over a field k is called *quasi-matched* (resp. *matched*) if there exist linear maps $\alpha : A \otimes X \longrightarrow X$ and $\beta : A \otimes X \longrightarrow A$ which turn X into a left A-quasi-module-coalgebra (resp. a left A-module-coalgebra) and A into a right X-quasi-module-coalgebra (resp. a right X-module-coalgebra), such that if one sets $\alpha(a \otimes x) = a \triangleright x$, $\beta(a \otimes x) = a \triangleleft x$ then the following conditions are satisfied:

$$a \triangleright (xy) = \sum_{(a)(x)} (a' \triangleright x')((a'' \triangleleft x'') \triangleright y), \tag{II.1}$$

$$a \triangleright 1 = \varepsilon(a)1, \tag{II.2}$$

$$(ab) \triangleleft x = \sum_{(b)(x)} (a \triangleleft (b' \triangleright x'))(b'' \triangleleft x''), \tag{II.3}$$

$$1 \triangleleft x = \varepsilon(x)1, \tag{II.4}$$

$$\sum_{(a)(x)} (a' \triangleleft x') \otimes (a'' \triangleright x'') = \sum_{(a)(x)} (a'' \triangleleft x'') \otimes (a' \triangleright x'), \tag{II.5}$$

for all a, $b \in A$ and x, $y \in X$, where 1 is the identity of X and of A respectively in (II.2) and (II.4).

For a quasi-matched (resp. matched) pair of bialgebras (X, A), we know from [1] and [2] that there exists an almost bialgebra (resp. a bialgebra) structure on the vector space $X \otimes A$ with identity equal to $1 \otimes 1$ such that its product is given by

$$(x \otimes a)(y \otimes b) = \sum_{(a)(y)} x(a' \triangleright y') \otimes (a'' \triangleleft y'')b, \tag{II.6}$$

its coproduct by

$$\Delta(x \otimes a) = \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'') \tag{II.7}$$

and its counit by

$$\varepsilon(x \otimes a) = \varepsilon_X(x)\varepsilon_A(a) \tag{II.8}$$

for all x, $y \in X$, a, $b \in A$. Equipped with this almost bialgebra (resp. bialgebra) structure, $X \otimes A$ is called the *quasi-bicrossed product* (resp. *bicrossed product*) of X and A, and denoted as $X \infty A$. Furthermore, the injective maps $i_X(x) = x \otimes 1$ and $i_A(a) = 1 \otimes a$ from X and A respectively into $X \infty A$ are bialgebra morphisms. Also, $x \infty a = (x \infty 1)(1 \infty a)$ for $a \in A$ and $x \in X$.

III Biperfect Weak Hopf Algebras And Quasi-Bicrossed Products

Definition III.1 *A weak Hopf algebra H is called: (i) a* perfect weak Hopf algebra *[7] if its weak antipode T is an anti-bialgebra morphism satisfying $(id * T)(H) \subseteq C(H)$ (the center of H); (ii) a* coperfect weak Hopf algebra *if its weak antipode T is an anti-bialgebra morphism satisfying $\sum_{(x)} x'T(x'') \otimes x''' = \sum_{(x)} x''T(x''') \otimes x'$ for any $x \in H$; (iii) a* biperfect weak Hopf algebra *if it is perfect and also coperfect.*

From Proposition 1.2 in [1], we know that if the weak antipode T of a weak Hopf algebra $H = (H, m, u, \Delta, \varepsilon, T)$ is an invertible anti-algebra morphism, then $H^{op} = (H, m^{op}, u, \Delta, \varepsilon)$ and $H^{cop} = (H, m, u, \Delta^{op}, \varepsilon)$ are both weak Hopf algebras with weak antipode T^{-1}.

Lemma III.1 *Suppose that $H = (H, m, u, \Delta, \varepsilon, T)$ is a weak Hopf algebra with T invertible, then H is perfect (resp. coperfect) if and only if H^{op} (resp. H^{cop}) is also perfect (resp. coperfect).*

Proof: When H is coperfect, then $\sum_{(x)} x'T(x'') \otimes x''' = \sum_{(x)} x''T(x''') \otimes x'$ for any $x \in H$. Thus,

$$(T \otimes 1) \sum_{(x)} (x''T^{-1}(x') \otimes x''') = (T \otimes 1) \sum_{(x)} (x'''T^{-1}(x'') \otimes x').$$

It follows that $\sum_{(x)} (x''T^{-1}(x') \otimes x''') = \sum_{(x)} (x'''T^{-1}(x'') \otimes x')$ since T is invertible. This means that H^{cop} is coperfect on the weak antipode T^{-1}. It is similar to prove the result in the case that H is perfect. #

For a finite dimensional weak Hopf algebra $H = (H, m, u, \Delta, \varepsilon, T)$, we know [1] that $H^* = (H^*, \Delta^*, \varepsilon^*, m^*, u^*, T^*)$ is a weak Hopf algebra with weak antipode T^*.

Lemma III.2 *A finite dimensional weak Hopf algebra H is perfect (resp. coperfect) if and only if its duality H^* is coperfect (resp. perfect).*

Proof: "only if": When H is perfect, we need to prove that for $f \in H^*$,

$$\sum_{(f)} f'T^*(f'') \otimes f''' = \sum_{(f)} f''T^*(f''') \otimes f'.$$

In fact, for a, $b \in H$,

$$\begin{aligned}
(\sum_{(f)} f'T^*(f'') \otimes f''')(a \otimes b) &= \sum_{(f)} (f'T^*(f''))(a) f'''(b) = \sum_{(f)(a)} f'(a')T^*(f'')(a'')f'''(b) \\
&= \sum_{(f)(a)} f'(a')f''(T(a''))f'''(b) = \sum_{(a)} f(a'T(a'')b) \\
&= \sum_{(a)} f(ba'T(a'')) = \sum_{(f)(a)} f'(b)f''(a')f'''(T(a'')) \\
&= \sum_{(f)(a)} f'(b)f''(a')T^*(f''')(a'') = \sum_{(f)} f'(b)(f''T^*(f'''))(a) \\
&= \sum_{(f)} (f''T^*(f''') \otimes f')(a \otimes b).
\end{aligned}$$

When H is coperfect, we need to prove that $(id_{H^*} * T^*)(H^*) \subseteq C(H^*)$.

In fact, for $f,\, g \in H^*$, $x \in H$,

$$\begin{aligned}(g(id_{H^*} * T^*)(f))(x) &= \sum_{(x)}(g(x')(id_{H^*} * T^*)(f))(x'') = \sum_{(x)} g(x')\Delta^*(id_{H^*} \otimes T^*)m^*(f)(x'') \\ &= \sum_{(x)} g(x')(id_{H^*} \otimes T^*)m^*(f)(x'' \otimes x''') = \sum_{(x)} g(x')m^*(f)(x'' \otimes T(x''')) \\ &= \sum_{(x)} g(x')f(x''T(x''')) = \sum_{(x)} f(x'T(x''))g(x''') \\ &= ((id_{H^*} * T^*)(f)g)(x).\end{aligned}$$

It is easy to see that T^* is an anti-bialgebra morphism from the same fact of T.

"if": It follows from $H \cong H^{**}$. #

Corollary III.3 *A finite dimensional weak Hopf algebra H is biperfect if and only if its duality H^* is biperfect.*

Lemma III.4 *Suppose that $H = (H, m, u, \Delta, \varepsilon, T)$ is a finite dimensional weak Hopf algebra and its weak antipode T is an invertible anti-bialgebra morphism. Then (i) H is perfect if and only if $(T * id)(H) \subseteq C(H)$; (ii) H is coperfect if and only if $\sum_{(x)} T(x')x'' \otimes x''' = \sum_{(x)} T(x'')x''' \otimes x'$ for any $x \in H$.*

Proof: (i) follows from Lemma 1.1 in [1].

(ii) Similar to the proof of Lemma III.2, we can prove that H satisfies $(T * id)(H) \subseteq C(H)$ if and only if H^* satisfies $\sum_{(f)} T^*(f')f'' \otimes f''' = \sum_{(f)} T^*(f'')f''' \otimes f'$ for $f \in H^*$.

Then H is coperfect if and only if H^* is perfect, if and only if $(T^* * id)(H^*) \subseteq C(H^*)$. But $H \cong (H^*)^*$. So, if and only if $\sum_{(x)} T(x')x'' \otimes x''' = \sum_{(x)} T(x'')x''' \otimes x'$ for $x \in H$. #

The concept of a Hopf pair of Hopf algebras was introduced by M.Takeuchi in [4], which plays a valid role in the study of the theory of quantum groups. Now, we generalize this and introduce some similar concepts corresponding to the weak Hopf algebras.

Definition III.2 *(i) Suppose that A and X are weak Hopf algebras with weak antipodes S_A and S_X, respectively. We call (X, A) a weak Hopf pair, if there exists a non-singular bilinear form $<,>$ from $X \otimes A$ to k satisfying*

$$< x, ab >= \sum_{(x)} < x', a >< x'', b >, \tag{III.1}$$

$$< x, 1_A >= \varepsilon(x), \tag{III.2}$$

$$< xy, a >= \sum_{(a)} < x, a' >< y, a'' >, \tag{III.3}$$

$$< 1_X, a >= \varepsilon(a), \tag{III.4}$$

$$< S_X(x), a >=< x, S_A(a) >, \tag{III.5}$$

where $x,\, y \in X$, $a,\, b \in A$.

(ii) In (i), moreover, if S_A is invertible and (III.1) and (III.5) are replaced with the following (III.6) and (III.7):

$$< x, ab >= \sum_{(x)} < x'', a >< x', b >, \tag{III.6}$$

$$< S_X(x), a >=< x, S_A^{-1}(a) >, \tag{III.7}$$

respectively, we call (X, A) a weak Hopf skew-pair*.*

From [1], $A^{op} = (A, \mu^{op}, \eta, \Delta, \varepsilon, S_A^{-1})$ is a weak Hopf algebra when S_A is invertible. Therefore (X, A) is a weak Hopf skew-pair if and only if (X, A^{op}) is a weak Hopf pair in the case where S_A is invertible.

We know from [3] that for two perfect weak Hopf algebras A and X with weak antipodes S_A and S_X respectively, suppose that A is cocommutative, S_A is invertible and (X, A) is a weak Hopf skew-pair, then (X, A) is a quasi-matched pair of bialgebra. We want to generalize this result to the case that A is non-cocommutative. In fact, we have the following lemma:

Lemma III.5 *For two perfect weak Hopf algebras A and X with weak antipodes S_A and S_X respectively, suppose that S_A is invertible and (X, A) is a weak Hopf skew-pair. Then A and X are both biperfect.*

Proof: For $x \in X$, $a, b \in A$, since A is perfect, we have

$$\begin{aligned}
\sum_{(x)} < x' S_X(x''), a >< x''', b > &= \sum_{(x)(a)} < x', a' >< S(x''), a'' >< x''', b > \\
&= \sum_{(x)(a)} < x', a' >< x'', S_A^{-1}(a'') >< x''', b > \\
&= \sum_{(a)} < x, b S_A^{-1}(a'') a' >= \sum_{(a)} < x, S_A^{-1}(a'') a' b > \\
&= \sum_{(x)(a)} < x', b >< x'', a' >< x''', S_A^{-1}(a'') > \\
&= \sum_{(x)(a)} < x', b >< x'', a' >< S_X(x'''), a'' > \\
&= \sum_{(x)} < x'' S_X(x'''), a >< x', b >,
\end{aligned}$$

and hence $\sum_{(x)}(x' S_X(x'') \otimes x''') = \sum_{(x)}(x'' S_X(x''') \otimes x')$. It means that X is coperfect.

Similarly, for x, $y \in X$, $a \in A$, since X is perfect, we can prove

$$\sum_{(a)} < x, a'' S_A^{-1}(a') >< y, a''' >= \sum_{(a)} < x, a''' S_A^{-1}(a'') >< y, a' > .$$

Hence $\sum_{(a)}(a'' S_A^{-1}(a') \otimes a''') = \sum_{(a)}(a''' S_A^{-1}(a'') \otimes a')$. Thus $\sum_{(a)}(a' S_A(a'') \otimes a''') = \sum_{(a)}(a'' S_A(a''') \otimes a')$. It follows that A is coperfect. #

Theorem III.6 *For two perfect weak Hopf algebras A and X with weak antipodes S_A and S_X respectively, suppose that S_A is invertible and (X, A) is a weak Hopf skew-pair. Then (X, A) is a quasi-matched pair of bialgebras with*

$$a \triangleright x = \sum_{(x)} < x' S_X(x'''), a > x'',$$

$$a \triangleleft x = \sum_{(a)} < x, S_A^{-1}(a''')a' > a''$$

so as to get a quasi-bicrossed product $X \infty A$, denoted as $D(X, A)$.

Proof: Firstly, we can verify easily the following:

$$< a \triangleright x, b >= \sum_{(a)} < x, S_A^{-1}(a'')ba' >, \tag{III.8}$$

$$< y, a \triangleleft x >= \sum_{(x)} < x' y S_X(x''), a > \tag{III.9}$$

for $a,\ b \in A,\ x,\ y \in X$.

Now we prove that A and X are a right X-quasi-module coalgebra and a left A-quasi-module coalgebra with the action $\triangleleft$ and $\triangleright$, respectively. In fact, for any $a \in A,\ x,\ y,\ z \in X$, we have

$$\begin{aligned} < z, a \triangleleft (xy) > &= \sum_{(xy)} < (xy)'' z S_X((xy)'), a > \\ &= \sum_{(x)(y)} < x'' y'' z S_X(y') S_X(x'), a >=< z, (a \triangleleft x) \triangleleft y >, \end{aligned}$$

then $a \triangleleft (xy) = (a \triangleleft x) \triangleleft y$; $< z, a \triangleleft 1 >=< 1 z S_X(1), a >=< z, a >$, then $a \triangleleft 1 = a$. Thus A is a right X-module. On the other hand,

$$\begin{aligned} < y \otimes z, \sum_{(a)(x)} (a' \triangleleft x') \otimes (a'' \triangleleft x'') > &= \sum_{(a)(x)} < y, a' \triangleleft x' >< z, a'' \triangleleft x'' > \\ &= \sum_{(a)(x)} < x' y S_X(x''), a' >< x''' z S_X(x^{(4)}), a'' > \\ &= \sum_{(x)} < x' y S_X(x'') x''' z S_X(x^{(4)}), a > \\ &= \sum_{(x)} < x' y z S_X(x''), a > \\ &= < yz, a \triangleleft x >=< y \otimes z, \Delta(a \triangleleft x) >, \end{aligned}$$

then $\Delta(a \triangleleft x) = \sum_{(a)(x)} (a' \triangleleft x') \otimes (a'' \triangleleft x'')$. It means that A is a right X-quasi-module-coalgebra. Similarly, we get that X is a left A-quasi-module-coalgebra.

Moreover, we can see that (II.2) and (II.4) are trivial according to (III.4) and (III.2) and the definition of $\triangleright$ and $\triangleleft$. And, using of Lemma III.5, we have

$$< \sum_{(a)(x)} (a' \triangleright x')((a'' \triangleleft x'') \triangleright y), b > \;=\; < \sum_{(a)(x)} (a' \triangleright x')(< x'', S_A^{-1}(a^{(4)})a'' > a''' \triangleright y), b >$$

$$
\begin{aligned}
&= \sum_{(a)(b)(x)} < x'', S_A^{-1}(a^{(4)})a'' >< (a' \triangleright x'), b' >< (a''' \triangleright y), b'' > \\
&= \sum_{(a)(b)(x)} < x'', S_A^{-1}(a^{(5)})a''' > \\
&\qquad < x', S_A^{-1}(a'')b'a' >< (a^{(4)} \triangleright y), b'' > \\
&= \sum_{(a)(b)} < x, S_A^{-1}(a^{(5)})a''' S_A^{-1}(a'')b'a' >< (a^{(4)} \triangleright y), b'' > \\
&= \sum_{(a)(b)} < x, S_{(A)}^{-1}(a''')b'a' >< (a'' \triangleright y), b'' > \\
&= \sum_{(a)(b)} < x, S_A^{-1}(a^{(4)})b'a' >< y, S_A^{-1}(a''')b''a'' > \\
&= \sum_{(a)} < xy, S_A^{-1}(a'')ba' >=< a \triangleright (xy), b >,
\end{aligned}
$$

then $a \triangleright (xy) = \sum_{(a)(x)} (a' \triangleright x')((a'' \triangleleft x'') \triangleright y)$, i.e. eq.(II.1) holds. Similarly, we get that $(ab) \triangleleft x = \sum_{(b)(x)} (a \triangleleft (b' \triangleright x'))(b'' \triangleleft x'')$, i.e. eq.(II.3) holds. Moreover,

$$
\begin{aligned}
\sum_{(a)(x)} (a' \triangleleft x') \otimes (a'' \triangleright x'') &= \sum_{(a)(x)} < x', S_A^{-1}(a''')a' > a'' \otimes < x'' S_X(x^{(4)}), a^{(4)} > x''' \\
&= \sum_{(a)(x)} < x', S_A^{-1}(a''')a' >< x'', a^{(4)} >< S_X(x^{(4)}), a^{(5)} > a'' \otimes x''' \\
&= \sum_{(a)(x)} < x', a^{(4)} S_A^{-1}(a''')a' >< S_X(x'''), a^{(5)} > a'' \otimes x'' \\
&= \sum_{(a)(x)} < x', a''' S_A^{-1}(a'')a' >< S_X(x'''), a^{(5)} > a^{(4)} \otimes x'' \\
&= \sum_{(a)(x)} < x', a' >< S_X(x'''), a''' > a'' \otimes x'' \\
&= \sum_{(a)(x)} < x', a' >< x''', S_A^{-1}(a''') > a'' \otimes x'' \\
&= \sum_{(a)(x)} < x', a' >< x''', S_A^{-1}(a^{(5)})a^{(4)} S_A^{-1}(a''') > a'' \otimes x'' \\
&= \sum_{(a)(x)} < x', a' >< x''', S_A^{-1}(a^{(5)})a''' S_A^{-1}(a'') > a^{(4)} \otimes x'' \\
&= \sum_{(a)(x)} < x', a' >< x''', S_A^{-1}(a'') >< x^{(4)}, S_A^{-1}(a^{(5)})a''' > a^{(4)} \otimes x'' \\
&= \sum_{(a)(x)} < x', a' >< S_X(x'''), a'' >< x^{(4)}, S_A^{-1}(a^{(5)})a''' > a^{(4)} \otimes x'' \\
&= \sum_{(a)(x)} < x' S_X(x'''), a' >< x^{(4)}, S_A^{-1}(a^{(4)})a'' > a''' \otimes x'' \\
&= \sum_{(a)(x)} (a'' \triangleleft x'') \otimes (a' \triangleright x'),
\end{aligned}
$$

then eq.(II.5) holds. In a word, (X, A) is a quasi-matched pair of bialgebras. Hence we get a quasi-bicrossed product $X \infty A$, denoted also as $D(X, A)$. #

Note that in this theorem, A is not required to be cocommutative. So, this theorem is a big improvement on the result obtained in [3].

IV Quantum Doubles Of Biperfect Weak Hopf Algebras

In Theorem III.6, when $A = H$ is a finite dimensional biperfect weak Hopf algebra with invertible weak antipode T, we set $X = H^{*cop}$ and suppose that $<,>$ is the bilinear form of H and its dual H^* as linear spaces. It was known in [1] that $(H^{*cop} = (H^*, \Delta^*, \varepsilon^*, (m^*)^{op}, u^*, (T^{*-1})$. It is easy to see that (H^{*cop}, H) is a weak Hopf skew-pair. Then (H^{*cop}, H) is a quasi-matched pair of bialgebras with $a \triangleright f = \sum_{(f)} < f' T^{*-1}(f'''), a > f''$ and $a \triangleleft f = \sum_{(a)} < f, T^{-1}(a''')a' > a''$ for $a \in H$ and $f \in H^{*cop}$ so as to get a quasi-bicrossed product $D(H^{*cop}, H) = H^{*cop} \infty H$, denoted briefly as $D(H)$ and called the *quantum double* of H.

Proposition IV.1 *Let $H = (H, m, u, \Delta, \varepsilon, T)$ be a finite dimensional biperfect weak Hopf algebra with invertible T. Then the multiplication in $D(H) = H^{*cop} \infty H$ is given by*

$$(f \infty a)(g \infty b) = \sum_{(a)} fg(T^{-1}(a''')?a') \infty a'' b$$

for f, $g \in H^{*cop}$, a, $b \in H$.

Proof:

$$\begin{aligned} (f \infty a)(g \infty b) &= \sum_{(a)(g)} f(a' \triangleright g') \infty (a'' \triangleleft g'')b \\ &= \sum_{(a)(g)} fg'(T^{-1}(a'')?a') \infty g''(T^{-1}(a^{(5)})a''')a^{(4)}b \\ &= \sum_{(a)} fg(T^{-1}(a^{(5)})a''' T^{-1}(a'')?a') \infty a^{(4)}b \\ &= \sum_{(a)} fg(T^{-1}(a^{(5)})?a''' T^{-1}(a'')a') \infty a^{(4)}b = \sum_{(a)} fg(T^{-1}(a''')?a') \infty a'' b. \quad \# \end{aligned}$$

Now we have the following main result:

Theorem IV.2 *Let $H = (H, m, u, \Delta, \varepsilon, T)$ be a finite dimensional biperfect weak Hopf algebra with invertible T. Then the quantum double $D(H)$ of H is quasi-braided equipped with a quasi-R-matrix $R = \sum_{i \in I} (1 \infty e_i) \otimes (e^i \infty 1) \in D(H) \otimes D(H)$ where $\{e_i\}_{i \in I}$ is a basis of the k-vector space H together with its dual basis $\{e^i\}_{i \in I}$ in H^{*cop}. Hence R is a solution of the quantum Yang-Baxter equation.*

Proof: For $f \in H^{*cop}$, $a \in H$,

$$\begin{aligned} \Delta^{op}(f \infty a)R &= \sum_{i \in I} \sum_{(f)(a)} (f'' \infty a'')(1 \infty e_i) \otimes (f' \infty a')(e^i \infty 1) \\ &= \sum_{i \in I} \sum_{(f)(a)} (f'' \varepsilon(T^{-1}(a^{(6)})?a^{(4)}) \infty a^{(5)} e_i) \otimes (f' e^i (T^{-1}(a''')?a') \infty a'') \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in I}\sum_{(f)(a)} (f''\varepsilon(T^{-1}(a^{(6)}))\varepsilon(a^{(4)})\varepsilon\infty a^{(5)}e_i)\otimes(f'e^i(T^{-1}(a''')?a')\infty a'')\\
&= \sum_{i\in I}\sum_{(f)(a)} (f''\infty a^{(4)}e_i)\otimes(f'e^i(T^{-1}(a''')?a')\infty a'');
\end{aligned}
$$

$$
\begin{aligned}
R\Delta(f\infty a) &= \sum_{i\in I}\sum_{(f)(a)} (\varepsilon\infty e_i)(f'\infty a')\otimes(e^i\infty 1)(f''\infty a'')\\
&= \sum_{i\in I}\sum_{(f)(a)(e_i)} (\varepsilon f'(T^{-1}(e_i''')?e_i')\infty e_i''a')\otimes(e^if''(T^{-1}(1)?1)\infty 1a'')\\
&= \sum_{i\in I}\sum_{(f)(a)(e_i)} (f'(T^{-1}(e_i''')?e_i')\infty e_i''a')\otimes(e^if''\infty a'').
\end{aligned}
$$

For every $b,\ c\in H,\ u,\ v\in H^{op*}$, let $\xi = b\otimes u\otimes c\otimes v$. Then

$$
\begin{aligned}
<\Delta^{op}(f\infty a)R,\xi> &= \sum_{i\in I}\sum_{(f)(a)} f''(b)u(a^{(4)}e_i)(f'e^i(T^{-1}(a''')?a'))(c)v(a'')\\
&= \sum_{i\in I}\sum_{(f)(a)(c)} f''(b)u(a^{(4)}e_i)f'(c')e^i(T^{-1}(a''')c''a')v(a'')\\
&= \sum_{i\in I}\sum_{(f)(a)(c)} f''(b)u(a^{(4)}e^i(T^{-1}(a''')c''a')e_i)f'(c')v(a'')\\
&= \sum_{(a)(c)} f(bc')u(a^{(4)}T^{-1}(a''')c''a')v(a'')\\
&= \sum_{(a)(c)} f(bc')u(c''a^{(4)}T^{-1}(a''')a')v(a'')\\
&= \sum_{(a)(c)} f(bc')u(c''a'''T^{-1}(a'')a')v(a^{(4)}) = \sum_{(a)(c)} f(bc')u(c''a')v(a'');
\end{aligned}
$$

$$
\begin{aligned}
<R\Delta(f\infty a),\xi> &= \sum_{i\in I}\sum_{(f)(a)(e_i)} f'(T^{-1}(e_i''')be_i')u(e_i''a')(e^if'')(c)v(a'')\\
&= \sum_{i\in I}\sum_{(f)(a)(e_i)(c)} f'(T^{-1}(e_i''')be_i')u(e_i''a')e^i(c')f''(c'')v(a'')\\
&= \sum_{i\in I}\sum_{(a)(e_i)(c)} f(c''T^{-1}(e_i''')be_i')u(e_i''a')e^i(c')v(a'')\\
&= \sum_{(a)(c)} f(c^{(4)}T^{-1}(c''')bc')u(c''a')v(a'')\\
&\quad (\text{since } \sum_{i\in I}\sum_{(e_i)} e^i(c)e_i'\otimes e_i''\otimes e_i''' = \sum_{(c)} c'\otimes c''\otimes c''')\\
&= \sum_{(a)(c)} f(c'''T^{-1}(c'')bc')u(c^{(4)}a')v(a'')\\
&= \sum_{(a)(c)} f(bc'''T^{-1}(c'')c')u(c^{(4)}a')v(a'')\\
&= \sum_{(a)(c)} f(bc')u(c''a')v(a'') = <\Delta^{op}(f\infty a)R,\xi>.
\end{aligned}
$$

Therefore $\Delta^{op}(f\infty a)R = R\Delta(f\infty a)$. Then $H^{*cop}\infty H$ is an almost quasi-cocommutative almost bialgebra with a universal quasi-R-matrix R.

We can prove in a way similar to the proof of Theorem 2.11 in [1] that

$$(\Delta \otimes id_H)(R) = R_{13}R_{23}; (id_H \otimes \Delta)(R) = R_{13}R_{12}.$$

It means that $H^{op*}\infty H$ is quasi-braided. Thus, by Proposition 2.8 in [1], R is a quasi-R-matrix. #

Note that since a cocommutative weak Hopf algebra must be coperfect, it means that Theorem IV.2 is a generalization of the one in [1] on the quantum double of a finite dimensional cocommutative perfect weak Hopf algebra.

It is easy to see that for a finite Clifford monoid $S = \{s_1, ..., s_n\}$ (see [8] and [1]), kS is a finite dimensional biperfect weak Hopf algebra with invertible weak antipode T_S satisfying $T_S(s) = s^{-1}$ for $s \in S$. Then by Theorem IV.2, the quantum double $D(kS)$ is quasi-braided equipped with a quasi-R-matrix $R = \sum_{i=1}^{n}(1\infty s_i) \otimes (s_i^*\infty 1) \in D(kS) \otimes D(kS)$ where s_i^* is the duality of s_i in $(kS)^{*cop}$. Thus, R is a solution of the quantum Yang-Baxter equation. But, this is also an example of a quantum quasi-double in Theorem 2.11 of [1] constructed from a finite dimensional cocommutative perfect weak Hopf algebra. So, it is very necessary to find an example of the quantum double from a finite dimensional biperfect weak Hopf algebra with invertible weak antipode which is not cocommutative.

For two biperfect weak Hopf algebras H and K, it is easy to prove that the tensor product $H \otimes K$ is also a biperfect weak Hopf algebra with the comultiplication $\Delta = (I \otimes T \otimes I)(\Delta_H \otimes \Delta_K)$, the multiplication $m = (m_H \otimes m_K)(I \otimes T \otimes I)$, the unit $1 = 1_H \otimes 1_K$, the counit $\varepsilon = \varepsilon_H \otimes \varepsilon_K$ and the weak antipode $T = T_H \otimes T_K$. $H \otimes K$ is commutative (resp. cocommutative) if and only if H and K are so.

For a finite non-commutative Clifford monoid S, let $H = kS$ with the weak antipode T_S, then $K = (kS)^*$ is also a finite dimensional biperfect weak Hopf algebra with invertible weak antipode T_S^*. Thus, we get a finite dimensional biperfect weak Hopf algebra $A = kS \otimes (kS)^*$ with invertible weak antipode $T = T_S \otimes T_S^*$, which is indeed not a Hopf algebra unless S is a group. Since kS is non-commutative, $(kS)^*$ is non-cocommutative. Hence A is non-commutative and non-cocommutative. By Theorem IV.2, the quantum double $D(A)$ of A is quasi-braided equipped with a quasi-R-matrix as a solution of the quantum Yang-Baxter equation. This construction is different from that of Theorem 2.11 in [1]. It implies that in Theorem IV.2, the quantum double of a finite dimensional biperfect weak Hopf algebra is indeed a generalization of that of a finite dimensional Hopf algebra.

We know in [2] that the R-matrix of quantum double of a finite dimensional Hopf algebra is invertible. But, for the quasi-R-matrix in Theorem IV.2, we can only get the regularity as following:

Proposition IV.3 *For a finite dimensional biperfect weak Hopf algebra H with invertible weak antipode T, the quasi-R-matrix $R = \sum_{i=1}^{n}(\varepsilon\infty e_i) \otimes (e_i^*\infty 1)$ of its quantum quasi-double $D(H)$ is a von Neumann regular element in $D(H) \otimes D(H)$ with its inverse $\bar{R} = \sum_{i=1}^{n}(\varepsilon\infty e_i) \otimes (e_i^* T\infty 1)$ where $\{e_1, ..., e_n\}$ is a basis of H and $\{e_1^*, ..., e_n^*\}$ is the dual basis in H^*.*

Proof: For any $\xi = b \otimes u \otimes c \otimes v \in H \otimes H^* \otimes H \otimes H^*$, we have

$$< R\bar{R}R, \xi > \;=\; \sum_{i,j,l} \varepsilon(b)u(e_ie_je_l)(e_i^*(e_j^*T)e_l^*)(c)v(1)$$

$$
\begin{aligned}
&= \varepsilon(b)v(1)\sum_{(c)} u(\sum_{i=1}^{n} e_i e_i^*(c') \sum_{j=1}^{n} e_j e_j^*(T(c'')) \sum_{l=1}^{n} e_l e_l^*(c''')) \\
&= \varepsilon(b)v(1)u(\sum_{(c)} c'T(c'')c''') = \varepsilon(b)v(1)u(c) \\
&= \varepsilon(b)v(1)\sum_{i=1}^{n} u(e_i)e_i^*(c) =< R, \xi >,
\end{aligned}
$$

then $R\bar{R}R = R$. Similarly, $< \bar{R}R\bar{R}, \xi > = < \bar{R}, \xi >$, then $\bar{R}R\bar{R} = \bar{R}$. #

For a left $D(H)$-module V, define $C^R_{V,V}$ satisfying $C^R_{V,V}(v \otimes w) = \tau(R(v \otimes w))$ for v, $w \in V$, then $C^R_{V,V}$ is a solution of the classical Yang-Baxter equation (see [7]) where τ is the flip map defined as $\tau(v_1 \otimes v_2) = v_2 \otimes v_1$. From Propositon IV.3, it is easy to prove that:

Corollary IV.4 *For a finite dimensional biperfect weak Hopf algebra H with invertible weak antipode T, let V be a left $D(H)$-module. Then $C^R_{V,V}$ is regular in the endomorphism monoid of $V \otimes V$, with its inverse $C^{\bar{R}}_{V,V}$ satisfying $C^{\bar{R}}_{V,V}(v \otimes w) = \tau(\bar{R}(v \otimes w))$ for v, $w \in V$.*

Now we discuss the representation-theoretic interpretation of $D(H)$.

Definition IV.1 *(see [2]) For a bialgebra H over k, a* crossed H-bimodule *V is a vector space together with linear maps $\mu_V : H \otimes V \longrightarrow V$ and $\Delta_V : V \longrightarrow V \otimes H$ such that*

(i) the map μ_V and Δ_V turn V into a left H-module and a right H-comodule respectively;

(ii) $\sum_{(a)(\beta)} a'\beta_V \otimes a''\beta_H = \sum_{(a)(a''\beta)} (a''\beta)_V \otimes (a''\beta)_H a'$ for all $a \in H$ and $\beta \in V$ where we set $\mu_V(a \otimes \beta) = a\beta$ and $\Delta_V(\beta) = \sum_{(\beta)} \beta_V \otimes \beta_H$.

Theorem IV.5 *Suppose H is a finite dimensional biperfect weak Hopf algebra with invertible weak antipode T. Then for a k-linear space V, the following statements are equivalent:*

(i) V is a left $D(H)$-module;

(ii) V is a crossed H-bimodule V and satisfies

$$
\sum_{(a)(\beta)} T^{-1}(a''')a''\beta_H \otimes a'\beta_V = \sum_{(\beta)} \beta_H \otimes a\beta_V \qquad \text{(IV.1)}
$$

for all $a \in H$ and $\beta \in V$.

Proof: "(i) $\Longrightarrow$ (ii)": Let V be a left $D(H)$-module. Since $H^{*cop} \cong H^{*cop}\infty 1$ and $H \cong \varepsilon\infty H$ are subalgebras of $D(H)$, V is a left H-module and also a left H^*-module with $a\beta = (1\infty a)\beta$ and $x\beta = (x\infty 1)\beta$ for $a \in H$, $x \in H^*$, $\beta \in V$. Then, $(ax)(\beta) = a(x\beta)$. But, by Proposition IV.1, we get

$$
\begin{aligned}
ax &= (1\infty a)(x\infty 1) = \sum_{(a)(x)} < x, T^{-1}(a''')?a' > \infty a'' \\
&= \sum_{(a)(x)} (< x, T^{-1}(a''')?a' > \infty 1)(1\infty a'').
\end{aligned}
$$

Hence, $a(x\beta) = \sum_{(a)(x)} < x, T^{-1}(a''')?a' > (a''\beta)$.

One must show that V can be endowed with a crossed H-bimodule structure. For μ_V, we define $\mu_V(a\otimes\beta)=a\beta$.

Given a basis $\{e_1,...,e_n\}$ of H and the dual basis $\{e_1^*,...,e_n^*\}$ of H^*, note that $x=\sum_{i=1}^n x(e_i)e_i^*$ and $a=\sum_{i=1}^n e_i^*(a)e_i$ for $x\in H^*$, $a\in H$.

Define $\Delta_V : V\longrightarrow V\otimes H$ satisfying $\Delta_V(\beta)=\sum_i e_i^*\beta\otimes e_i$ for any $\beta\in V$. Consider the dual Δ_V^* of Δ_V, we have that for any $\alpha\in V^*$, $\beta\in V$, $x\in H^*$,

$$\begin{aligned}<\Delta_V^*(\alpha\otimes x),\beta> &= <\alpha\otimes x,\Delta_V(\beta)>=\sum_{i=1}^n<\alpha,e_i^*\beta><x,e_i>\\ &= <\alpha,(\sum_{i=1}^n<x,e_i>e_i^*)\beta>=<\alpha,x\beta>;\end{aligned}$$

in particular, for $x=\varepsilon$ (the identity of H^*) and any $\beta\in V$,

$$<\Delta_V^*(\alpha\otimes\varepsilon),\beta>=<\alpha,\beta>.$$

Then $\Delta_V^*(\alpha\otimes\varepsilon)=\alpha$. It follows that, for any $y\in H^*$,

$$\begin{aligned}<\Delta_V^*(\Delta_V^*(\alpha\otimes x)\otimes y),\beta> &= <\Delta_V^*(\alpha\otimes x),y\beta>\\ &= <\alpha,x(y\beta)>=<\Delta_V^*(\alpha\otimes xy),\beta>,\end{aligned}$$

then $\Delta_V^*(\Delta_V^*(\alpha\otimes x)\otimes y)=\Delta_V^*(\alpha\otimes xy)$. Hence V^* is a right H^*-module under the action Δ_V^*. Therefore, dually, V becomes a right H-comodule under the coaction Δ_V.

For $a\in H$, $\beta\in V$, $x\in H^{*cop}$, we have

$$\begin{aligned}(id\otimes x)(\sum_{(a)(\beta)}a'\beta_V\otimes a''\beta_H) &= \sum_{(a)}\sum_{i=1}^n a'(e_i^*\beta)x(a''e_i)\\ &= \sum_{(a)(x)}\sum_{i=1}^n a'(e_i^*\beta)x'(e_i)x''(a'')=\sum_{(a)(x)}a'(x'\beta)x''(a'')\\ &= \sum_{(a)(x)}x''(a^{(4)})x'(T^{-1}(a''')?a')(a''\beta)\\ &= \sum_{(a)}x(a^{(4)}T^{-1}(a''')?a')(a''\beta)\\ &= \sum_{(a)}x(?a'''T^{-1}(a'')a')(a^{(4)}\beta)=\sum_{(a)}x(?a')(a''\beta)\\ &= \sum_{(a)(x)}x'(a')x''(a''\beta)=\sum_{(a)(x)}\sum_{i=1}^n x'(a')x''(e_i)e_i^*(a''\beta)\\ &= \sum_{(a)}\sum_{i=1}^n x(e_ia')e_i^*(a''\beta)=(id\otimes x)(\sum_{(a)}\sum_{i=1}^n e_i^*(a''\beta)\otimes e_ia')\\ &= (id\otimes x)(\sum_{(a''\beta)(a)}(a''\beta)_V\otimes(a''\beta)_Ha').\end{aligned}$$

It follows that

$$\sum_{(a)(\beta)}a'\beta_V\otimes a''\beta_H=\sum_{(a''\beta)(a)}(a''\beta)_V\otimes(a''\beta)_Ha'. \tag{IV.2}$$

Hence by Definition IV.1, V is a crossed H-bimodule.

Now, we prove the formula (IV.1). With the μ_V and Δ_V as defined above, we have proved that for any $\alpha \in V^*$, $\beta \in V$, $x \in H^*$, $< \Delta_V^*(\alpha \otimes x), \beta >=< \alpha, x\beta >$. But,

$$< \Delta_V^*(\alpha \otimes x), \beta >=< \alpha \otimes x, \beta_V \otimes \beta_H >=< \alpha, \beta_V >< x, \beta_H >=< \alpha, < x, \beta_H > \beta_V > .$$

So, it follows

$$x\beta =< x, \beta_H > \beta_V. \tag{IV.3}$$

And, since V is a $D(H)$-module, we have $a(x\beta) = (ax)\beta$. However, by the formula (IV.3), $a(x\beta) = \sum_{(\beta)} < x, \beta_H > a\beta_V$; and by the formulas (IV.3) and (IV.2),

$$\begin{aligned} (ax)\beta &= (\sum_{(a)} < x, T^{-1}(a''')?a' > \infty a'')\beta = (\sum_{(a)} < x, T^{-1}(a''')?a' > a'')\beta \\ &= \sum_{(a)} < x, T^{-1}(a''')?a' > (a''\beta) = \sum_{(a)(x)} < x''', T^{-1}(a''') >< x', a' > x''(a''\beta) \\ &= \sum_{(a)(x)(a''\beta)} < x''', T^{-1}(a''') >< x'', (a''\beta)_H >< x', a' > (a''\beta)_V \\ &= \sum_{(a)(a''\beta)} < x, T^{-1}(a''')(a''\beta)_H a' > (a''\beta)_V \\ &= \sum_{(a)(\beta)} < x, T^{-1}(a''')a''\beta_H > a'\beta_V. \end{aligned}$$

Hence, for any $x \in H^{*cop}$, $\sum_{(\beta)} < x, \beta_H > a\beta_V = \sum_{(a)(\beta)} < x, T^{-1}(a''')a''\beta_H > a'\beta_V$. Then the formula (IV.1) follows.

"$(ii) \Longrightarrow (i)$": Say that V is a crossed H-bimodule about μ_V and Δ_V. Then, V is a left H-module about μ_V and a right H-comodule about Δ_V. Write $\mu_V(a \otimes \beta) = a\beta$ for $a \in H$, $\beta \in V$. For $x \in H^*$, $\beta \in V$, let $x\beta = \sum_{(\beta)} < x, \beta_H > \beta_V$, where $\Delta_V(\beta) = \sum_{(\beta)} \beta_V \otimes \beta_H$. Since Δ_V is a right coaction, it is easy to show that $(xy)\beta = x(y\beta)$ for $y \in H^*$. Then it follows that V is a left H^{*cop}-module.

Set $(xa)\beta = x(a\beta)$ for $x \in H^{*cop}$, $a \in H$, $\beta \in V$. Then, by (IV.1),

$$a(x\beta) = \sum_{(\beta)} < x, \beta_H > a\beta_V = \sum_{(a)(\beta)} < x, T^{-1}(a''')a''\beta_H > a'\beta_V = (ax)\beta.$$

where the first equality follows from (VI.3), the second from (VI.1) and the third from (VI.3) and (VI.2) as proved in "$(i) \Longrightarrow (ii)$".

Therefore, V becomes a left $D(H)$-module since H and H^{*cop} are subalgebras of $D(H)$ and the multiplication of $D(H)$ is determined by the interaction of H and H^{*cop}. #

V Examples From Matrix Groups

Now, we give some examples from a concrete Clifford monoid. The definition of a Clifford semigroup/monoid can be found in [8] and [1].

Let $Y = \{\alpha, \beta, \gamma, \rho, \sigma, \delta\}$ be the semilattice with multiplication "$\cdot$" given by the following table:

$\cdot$	α	β	γ	ρ	σ	δ
α	α	α	α	α	α	α
β	α	β	β	α	β	β
γ	α	β	γ	α	β	γ
ρ	α	α	α	ρ	ρ	ρ
σ	α	β	β	ρ	σ	σ
δ	α	β	γ	ρ	σ	δ

The partial order in the semilattice Y can be presented as the diagram below:

$$\begin{tikzcd}
\delta \arrow[r] \arrow[d] & \sigma \arrow[r] \arrow[d] & \rho \arrow[d] \\
\gamma \arrow[r] & \beta \arrow[r] & \alpha
\end{tikzcd}$$

Obviously, δ is the identity of Y.

For a ring R with identity, $R^{2\times 2}$ denotes the 2×2 full matrix ring over R, $U(R)$ the group consisting of all units in R. Let Z be the integer number ring. For a prime number p, Z_p is a field and $U(Z_p^{2\times 2})$ is just the 2×2 general linear group $GL_2(Z_p)$ over Z_p. Assume that $G_\alpha=\{e_\alpha\}$ and $G_\delta=\{e_\delta\}$ are the trivial groups, $G_\beta=GL_2(Z_2)$, $G_\gamma=U(Z_4^{2\times 2})$, $G_\rho=GL_2(Z_3)$, $G_\sigma=U(Z_6^{2\times 2})$. Then $G_u\cap G_v=\emptyset$ for any $u,v\in Y$, $u\neq v$. Set $S=\cup_{u\in Y}G_u$. We will define a multiplication on S such that $S=\cup_{u\in Y}G_u$ becomes a Clifford monoid related to the semilattice Y.

Firstly, we mention the fact that over a commutative ring R with identity, an $m\times m$ matrix X is invertible if and only if $\det X$ is a unit in R.

Then, for $n=2,3,4,6$, $X=\begin{bmatrix} x & y\\ a & b\end{bmatrix}\in U(Z_n^{2\times 2})$ if and only if $\det X=xb-ay\in U(Z_n)$. It is easy to see $U(Z_6)=\{\bar{1},\bar{5}\}$, $U(Z_4)=\{\bar{1},\bar{3}\}$, $U(Z_3)=\{\bar{1},\bar{2}\}$, $U(Z_2)=\{\bar{1}\}$.

A ring homomorphism $\pi_{\sigma,\rho}:Z_6\longrightarrow Z_3$ can be defined which satisfies $\pi_{\sigma,\rho}(\bar{0})=\bar{0}$, $\pi_{\sigma,\rho}(\bar{1})=\bar{1}$, $\pi_{\sigma,\rho}(\bar{2})=\bar{2}$, $\pi_{\sigma,\rho}(\bar{3})=\bar{0}$, $\pi_{\sigma,\rho}(\bar{4})=\bar{1}$ and $\pi_{\sigma,\rho}(\bar{5})=\bar{2}$.

For $X=\begin{bmatrix} x & y\\ a & b\end{bmatrix}\in U(Z_6^{2\times 2})=G_\sigma$, we have $\det X=xb-ay=\bar{1}$, or $\bar{5}$, then

$$\pi_{\sigma,\rho}(x)\pi_{\sigma,\rho}(b)-\pi_{\sigma,\rho}(a)\pi_{\sigma,\rho}(y)=\bar{1},\ \mathit{or}\ \bar{2}.$$

It follows $\begin{bmatrix} \pi_{\sigma,\rho}(x) & \pi_{\sigma,\rho}(y)\\ \pi_{\sigma,\rho}(a) & \pi_{\sigma,\rho}(b)\end{bmatrix}\in GL_2(Z_3)=G_\rho$. Thus, we can expand $\pi_{\sigma,\rho}$ to make it a group homomorphism from G_σ to G_ρ. For this, it is enough to define $\pi_{\sigma,\rho}:G_\sigma\longrightarrow G_\rho$ satisfying

$$\pi_{\sigma,\rho}\begin{bmatrix} x & y\\ a & b\end{bmatrix}=\begin{bmatrix} \pi_{\sigma,\rho}(x) & \pi_{\sigma,\rho}(y)\\ \pi_{\sigma,\rho}(a) & \pi_{\sigma,\rho}(b)\end{bmatrix}$$

since $\pi_{\sigma,\rho}\Big(\begin{bmatrix} x_1 & y_1\\ a_1 & b_1\end{bmatrix}\begin{bmatrix} x_2 & y_2\\ a_2 & b_2\end{bmatrix}\Big)=\pi_{\sigma,\rho}\begin{bmatrix} x_1 & y_1\\ a_1 & b_1\end{bmatrix}\pi_{\sigma,\rho}\begin{bmatrix} x_2 & y_2\\ a_2 & b_2\end{bmatrix}$ can be shown easily using the fact that $\pi_{\sigma,\rho}$ is a ring homomorphism from Z_6 to Z_3.

Note that $\pi_{\sigma,\rho}(\bar{5}) = \bar{2} \in U(Z_3)$, so $\pi_{\sigma,\rho}$ is an epimorphism from G_σ to G_ρ.

Similarly, the ring homomorphisms $\pi_{\sigma,\beta} : Z_6 \longrightarrow Z_2$ and $\pi_{\gamma,\beta} : Z_4 \longrightarrow Z_2$ can be defined respectively satisfying $\pi_{\sigma,\beta}(\bar{0}) = \bar{0}$, $\pi_{\sigma,\beta}(\bar{1}) = \bar{1}$, $\pi_{\sigma,\beta}(\bar{2}) = \bar{0}$, $\pi_{\sigma,\beta}(\bar{3}) = \bar{1}$, $\pi_{\sigma,\beta}(\bar{4}) = \bar{0}$, $\pi_{\sigma,\beta}(\bar{5}) = \bar{1}$ and $\pi_{\gamma,\beta}(\bar{0}) = \bar{0}$, $\pi_{\gamma,\beta}(\bar{1}) = \bar{1}$, $\pi_{\gamma,\beta}(\bar{2}) = \bar{0}$, $\pi_{\gamma,\beta}(\bar{3}) = \bar{1}$. Moreover, the group homomorphisms $\pi_{\sigma,\beta} : G_\sigma \longrightarrow G_\beta$ and $\pi_{\gamma,\beta} : G_\gamma \longrightarrow G_\beta$ can be constructed in a similar way.

Finally, we define $\pi_{\beta,\alpha} : G_\beta \longrightarrow G_\alpha$, $\pi_{\rho,\alpha} : G_\rho \longrightarrow G_\alpha$, $\pi_{\delta,\sigma} : G_\delta \longrightarrow G_\sigma$, $\pi_{\delta,\gamma} : G_\delta \longrightarrow G_\gamma$ as the trivial group homomorphisms. Then one has the following diagram:

$$\begin{tikzcd}
G_\delta \arrow[r, "\pi_{\delta,\sigma}"] \arrow[d, "\pi_{\delta,\gamma}"] & G_\sigma \arrow[r, "\pi_{\sigma,\rho}"] \arrow[d, "\pi_{\sigma,\beta}"] & G_\rho \arrow[d, "\pi_{\rho,\alpha}"] \\
G_\gamma \arrow[r, "\pi_{\gamma,\beta}"] & G_\beta \arrow[r, "\pi_{\beta,\alpha}"] & G_\alpha
\end{tikzcd}$$

Now, we introduce the multiplication"$\cdot$" in S by $XW = \pi_{u,uv}(X)\pi_{v,uv}(W)$ if $X \in G_u$ and $W \in G_v$ for $u, v \in Y$. Then, with this multiplication, $S = \cup_{u\in Y} G_u$ becomes a Clifford monoid related to the semilattice Y, and the only element e_δ of G_δ is the identity of S.

Obviously, S is a finite and noncommutative Clifford monoid. Then for the cocommutative weak Hopf algebra kS we may obtain the quantum double $D(S)$ and its quasi-R-matrix R by using the result in [1]. We have the decomposition of linear spaces as follows:

$$D(S) = (kS)^{op*} \infty (kS) = (\oplus_{u\in Y} kG_u)^{op*} \infty (\oplus_{u\in Y} kG_u) = \oplus_{u,v\in Y}((kG_u)^{op*} \infty (kG_v))$$

where $(kG_u)^{op*}\infty(kG_v)$ means a direct summand of $D(S)$ and ∞ is same in $D(S)$ since $(kS)^{op*} = \oplus_{u\in Y}(kG_u)^{op*}$ and $kS = \oplus_{u\in Y} kG_u$ such that for each $u \in Y$, $(kG_u)^{op*}$ is embedded into $(kS)^{op*}$ and kG_u is embedded into kS.

Any $\varphi \in (kG_u)^{op*}$ can be expanded to $\overline{\varphi} \in (kS)^{op*}$ satisfying $\overline{\varphi}(U + V) = \varphi(U)$ for any element $U + V$ of $kS = \oplus_{v\in Y} kG_v$ where $U \in kG_u$ and $V \in \oplus_{v\neq u} kG_v$.

Let u_1, u_2, v_1, $v_2 \in Y$, $X \in G_{v_1}$, $W \in G_{v_2}$, $A \in G_{u_1}$, $B \in G_{u_2}$. Then their dual elements ϕ_A and ϕ_B of A and B are in $(kG_{u_1})^{op*}$ and $(kG_{u_2})^{op*}$, respectively, where $\phi_A(C) = \begin{cases} 0 & \text{if } C \in G_{u_1}, C \neq A \\ 1 & \text{if } C = A \end{cases}$ and ϕ_B is given similarly.

The multiplication of $D(S)$ can be presented by

$$(\phi_A \infty X)(\phi_B \infty W) = (\overline{\phi_A} \infty X)(\overline{\phi_B} \infty W) = \overline{\phi_A}\,\overline{\phi_B}(X^{-1}?X) \infty XW,$$

where

$$\overline{\phi_A}\,\overline{\phi_B}(X^{-1}?X) = \begin{cases} 0 & \text{if } X^{-1}AX \neq B \\ \overline{\phi_A} = \phi_A & \text{if } X^{-1}AX = B. \end{cases}$$

By [1], the quasi-R-matrix of $D(S)$ is

$$R = \sum_{s\in S} (1\infty s) \otimes_k (\overline{\phi_s} \infty 1) = \sum_{u\in Y} \sum_{g_u\in G_u} (1\infty g_u) \otimes_k (\phi_{g_u} \infty 1) \in D(S) \otimes D(S).$$

¿From $U(Z_6) = \{\bar{1}, \bar{5}\}$, $U(Z_4) = \{\bar{1}, \bar{3}\}$, $U(Z_3) = \{\bar{1}, \bar{2}\}$, $U(Z_2) = \{\bar{1}\}$ and the fact that $X = \begin{bmatrix} x & y \\ a & b \end{bmatrix} \in U(Z_n^{2\times 2})$ if and only if $\det X = xb - ay \in U(Z_n)$, it is easy to compute $|G_u|$ for

each $u \in Y$. We have $|G_\delta| = 1, |G_\alpha| = 1, |G_\beta| = 6, |G_\gamma| = 96, |G_\rho| = 48, |G_\sigma| = 288$. It follows that the number of monomials of R of $D(S)$ is $|S| = |G_\delta| + |G_\alpha| + |G_\beta| + |G_\gamma| + |G_\rho| + |G_\sigma| = 440$. Therefore, from the Clifford monoid $S = \cup_{u \in Y} G_u$ we have constructed an example of the quantum doubles of cocommutative weak Hopf algebras in [1].

In the following we give an example of the quantum doubles of perfect (noncocommutative) weak Hopf algebras.

For the Clifford monoid S above, $H = kS \otimes (kS)^*$ is a finite dimensional non-commutative and non-cocommutative biperfect weak Hopf algebra with invertible weak antipode $T = T_S \otimes T_S^*$ satisfying $T_S(X) = X^{-1}$ and $T_S^*(f)(X) = f(T_S(X))$ for any matrix $X \in S$ and $f \in (kS)^*$. The dimension of H is dim$H$ =dim$(kS \otimes (kS)^*) = |S|^2 = 193600$. The quantum double is given by

$$\begin{aligned} D(H) &= H^{op*} \infty H = (kS \otimes (kS)^*)^{op*} \infty (kS \otimes (kS)^*) \\ &= ((kS)^{op*} \otimes (kS)^{cop}) \infty (kS \otimes (kS)^*) = ((kS)^{op*} \otimes kS) \infty (kS \otimes (kS)^*) \\ &= \sum_{u,v,p,q \in Y} ((kG_u)^{op*} \otimes kG_v) \infty (kG_p \otimes (kG_q)^*), \end{aligned}$$

where $kG_p \otimes (kG_q)^*$ and $(kG_u)^{op*} \otimes kG_v$ are as subspaces of $kS \otimes (kS)^*$ and $(kS)^{op*} \otimes kS$. The multiplication of $D(S)$ obeys the formula in Proposition IV.1.

The quasi-R-matrix of $D(H)$ is

$$R = \sum_{p,\, q \in Y} \sum_{g_p \in G_p,\, g_q \in G_q} ((1 \otimes 1) \infty (g_p \otimes \phi_{g_q})) \otimes ((\phi_{g_p} \otimes g_q) \infty (1 \otimes 1)),$$

whose number of monomials is $|S|^2 = 193600$.

Acknowledgement: We would like to express our sincere thanks to the anonymous referee for detailed and valuable comments and suggestions which bring the paper to its present form.

This work is financially supported by Australian Research Council. Fang Li is also supported by the NSF of Zhejiang Province of China (Project No.102028) and a Raybould Fellowship from the University of Queensland.

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